SOME REMARKS ON THE EXISTENCE AND UNIQUENESS THEOREM (THEOREM 1.3.2)

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{1}$$

Theorem 1. *Assume that*

$$R = \{(x,y) : a \le x \le b, \quad c \le y \le d\}$$

is a rectangle containing (x_0, y_0)*. Suppose that*

(i) f *is continuous in* R*;*

(ii) $\frac{\partial f}{\partial y}$ *is continuous in* R*.*

Then (1) *has a unique solution on some interval* I *containing* x_0*.*

Remark 2.

1. *Condition (i) guarantees the existence of a solution.*
2. *If Condition (ii) fails, then* (1) *may have multiple solutions.*

Example 3.

$$\frac{dy}{dx} = \sqrt{y}, \quad y(0) = 0. \tag{2}$$

Solution. This is a separable DE. More precisely,

$$\int \frac{dy}{\sqrt{y}} = \int dx + C \Longrightarrow 2\sqrt{y} = x + C$$
$$\Longrightarrow y = (\frac{x+C}{2})^2.$$

Plugging in $y(0) = 0$, we obtain

$$0 = y(0) = C^2/4 \Longrightarrow C = 0 \Longrightarrow y = \frac{x^2}{4}.$$

However, one can verify without any difficulty that $y \equiv 0$ is a another solution to (2).

Does this example violates Theorem 1.3.2?

The answer is NO!

Let's check the two conditions in Theorem 1.3.2.

(i) $f(x,y) = \sqrt{y}$ is continuous in $\mathbb{R}^2$, thus in any rectangle containing $(0,0)$.

(ii) $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$ is NOT continuous on any rectangle containing $(0,0)$.

So Condition (i) is fulfilled, so there do exists some solutions. However, the failure of Condition (ii) tells us that the number of solutions to (2) is not necessarily one. Indeed, as we have shown, there are at least two solutions. ◀

Remark 4. *You may verify via Theorem 1.3.2 that*

$$\frac{dy}{dx} = \sqrt{y}, \quad y(0) = a \tag{3}$$

has a unique solution for any $a > 0$.